

The Chebotarev-Gregoratti Hamiltonian as singular perturbation of a nonsemibounded operator

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Abstract

We derive the Hamiltonian associated to a quantum stochastic flow by extending the Albeverio-Kurasov construction of self-adjoint extensions to finite rank perturbations of nonsemibounded operators to Fock space.

1 Introduction

To analyze a dynamical system with Hamiltonian $K = K_0 + \Upsilon$, occurring as a perturbation of a free Hamiltonian K_0 , we transfer to the interaction picture by means of the wave operators

$$V(t) = U_0(-t)U(t)$$

where U and U_0 are the strongly continuous groups generated by K and K_0 respectively. The family $V = \{V(t) : t \geq 0\}$ then satisfies the differential equation

$$\dot{V}(t) = -i\Upsilon(t)V(t)$$

where $\Upsilon(t) = U_0(-t)\Upsilon U_0(t)$. However, V is not a semi-group, but instead forms a U_0 -cocycle, that is $V(t+s) = U_0(-t)V(s)U_0(t)V(t)$ for all $t, s \geq 0$.

Conversely, given a strongly continuous group U_0 and a strongly continuous U_0 -cocycle V , then we may define a family U by

$$U(t) = \begin{cases} U_0(t)V(t), & t \geq 0; \\ V(-t)^\dagger U_0(t), & t < 0. \end{cases}$$

and this is readily seen to be a strongly continuous group. Consequently we deduce the existence of an associated Hamiltonian K .

The somewhat surprising feature of the converse is that V does not need to be strongly differentiable. In particular, it applies to the class of quantum

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stochastic evolutions modelling open system dynamics. In such cases, the existence of Hamiltonians K_0 and K is immediately apparent, however, it must of necessity be the case that domains do not have dense intersection and so the formal subtraction $\Upsilon = K - K_0$ does not lead to a self-adjoint operator in any meaningful way.

A well studied model is the dilations of an irreversible dynamics using Fock space over L^2 -functions of time where the free dynamics is second quantization of the time shift. It has been a long standing problem to characterize the associated Hamiltonian K for these models [1]. The major breakthrough came in 1997 when A.N. Chebotarev solved this problem for the class of quantum stochastic evolutions satisfying Hudson-Parthasarathy differential equations with bounded commuting system coefficients [7],[8]. His insight was based on scattering theory of a one-dimensional system with a Dirac potential, say, with formal Hamiltonian

$$k = i\partial + E\delta$$

describing a one-dimensional particle propagating along the negative x -axis with a delta potential of strength E at the origin. (In Chebotarev's analysis the δ -function is approximated by a sequence of regular functions, and a strong resolvent limit is performed.) The mathematical techniques used in this approach were subsequently generalized by Gregoratti [14] to relax the commutativity condition. More recently, the analysis has been further extended to treat unbounded coefficients [17].

Independently, several authors have been engaged in the program of describing the Hamiltonian nature of quantum stochastic evolutions by interpreting the time-dependent function $\Upsilon(t)$ as being an expression involving quantum white noises satisfying a singular CCR [9][10][11][3]. This would naturally suggest that Υ should be interpreted as a sesquilinear expression in these noises at time $t = 0$. Specifically one considered time dependent Wick ordered expressions $\Upsilon(t) = \sum_{ij} E_{ij} a_i^\dagger(t) a_j(t) + \sum_i E_{i0} a_i^\dagger(t) + \sum_j E_{0j} a_j(t) + E_{00}$ where the $E_{ij} = E_{ji}^\dagger$ are operators on the initial space and $a_i(t), a_i^\dagger(t)$ are delta-correlated quantum white noises corresponding to the formal derivatives of the annihilation and creation process. With $a_i(t) = U_0(-t) a_i(0) U_0(t)$, we see that $\Upsilon(t) \equiv U_0(-t) \Upsilon U_0(t)$ where $\Upsilon = \Upsilon(0)$. We shall show that this intuition essentially provides the correct answer, though without using quantum white noise! This approach inspired W. von Waldenfels to give an alternative construction of the associated Hamiltonian, for diffusions [18] and simple jump processes [19], however this was formulated through the conventions of kernel calculus.

The aim of the current paper is to complete this program by returning to the original one-dimensional model considered by Chebotarev. We alternatively consider this as a problem of finding a suitable self-adjoint for the singular Hamiltonian [16]. Here the generator of the free dynamics $k_0 = i\partial$ is not semi-bounded and the δ -perturbation is viewed as a singular rank-one perturbation. We employ methods introduced by Albeverio and Kurasov [4][5] [6] to construct self-adjoint extensions of such models. In particular, the boundary conditions

that one imposes at the origin corresponds to a phase change of $s = \frac{1-\frac{i}{2}E}{1+\frac{i}{2}E}$, which should be contrasted with the condition $s = e^{-iE}$ deduced by Chebotarev. We show that the constructions of Albeverio and Kurasov are amenable to second quantization and the form of Υ suggested by quantum white noise analysis is precisely what is needed to obtain the description of the associated Hamiltonian K derived by Chebotarev, Gregoratti and von Waldenfels. The construction avoids both the use of quantum white noises and the unwieldy complexity of the kernel calculus by instead using the defect vectors which lie in the underlying one-particle Hilbert space. One important subtlety is that the continuous tensor product decomposition for Fock spaces is here implemented by the Sobolev space $W^{1,2}$ inner product and not the usual L^2 inner product.

We show that there is a natural class of boundary conditions giving rise to different extensions, and therefore different physical representations of the same problem. These are parameterized by what effectively are the additional complex damping terms (describing for instance energy shifts, e.g., the Lamb shift [2]) and which have earlier been referred to as a degree of “gauge freedom”.

2 Singular Perturbations

We shall adopt the standard convention that the sesquilinear Hilbert space inner product is conjugate linear in the first argument and linear in the second. Let ξ be a conjugate linear functional on some domain of functions on the real line. Its adjoint ξ^\dagger is the linear functional on the same domain defined through complex conjugation, that is, $\xi^\dagger(\phi) = \xi(\phi)^*$. With a standard abuse of the Dirac bra-ket notation, we shall write $\xi^\dagger(\phi) = \langle \xi | \phi \rangle$, $\xi(\phi) = \langle \phi | \xi \rangle$, or more simply

$$\xi = |\xi\rangle, \quad \xi^\dagger = \langle \xi|,$$

even though the functionals need not correspond to vectors in the Hilbert space $L^2(\mathbb{R})$.

The weak derivative of a measurable function ϕ will be denoted as $\partial\phi$. For a fixed \mathfrak{H} be a Hilbert space and I an open subset of \mathbb{R}^d , we shall write $W^{1,2}(I, \mathfrak{H})$ for the Sobolev space of \mathfrak{H} -valued functions ϕ possessing a (weak) derivative and such that both ϕ and $\partial\phi$ are square-integrable. This is again a Hilbert space with inner product

$$\langle \phi | \psi \rangle_{1,2} \triangleq \int_I (\phi^* \psi + \partial\phi^* \partial\psi).$$

Note that the corresponding norm $\|\cdot\|_{1,2}$ is then the graph norm associated with the first derivative operator ∂ .

Formally one may consider the Hamiltonian

$$k = i\partial + E\delta$$

describing a one-dimensional particle propagating along the negative x -axis with a delta potential of strength E at the origin. As a mathematical problem, we

then have a perturbation of a self-adjoint operator $k_0 = i\partial$, which is not semi-bounded, by a rank-one perturbation $\Upsilon = E|\delta\rangle\langle\delta|$ - that is to say $\Upsilon\phi(x) = \phi(0)\delta(x)$. The Dirac functional δ is however bounded on the domain of k_0 . The singular nature of the potential implies that the particle wavefunction will be discontinuous at the origin. In particular, such functions will not be in the domain of the operator k_0 .

We now review the theory of self-adjoint extensions of the generator of linear translations $i\partial$ on the punctured line $\mathbb{R}/\{0\}$.

2.1 Distributions on discontinuous test functions

Let $AC(I)$ denote the set of absolutely continuous functions on an open subset I of the real line. We consider some singular functionals on this space.

Definition 1 *The one-sided delta functionals δ_\pm on $AC(\mathbb{R}/\{0\})$ are given by*

$$\langle\delta_\pm|\phi\rangle = \phi(0^\pm),$$

and we introduce the associated functionals:

$$\begin{aligned} (\text{jump at the origin}) \quad \langle j| &\triangleq \langle\delta_+ - \delta_-|, \\ (\text{symmetric delta functional}) \quad \langle\delta_\star| &\triangleq \tfrac{1}{2}\langle\delta_+ + \delta_-|. \end{aligned}$$

That is, $\langle j|\phi\rangle = \phi(0^+) - \phi(0^-)$ and $\langle\delta_\star|\phi\rangle = \tfrac{1}{2}\phi(0^+) + \tfrac{1}{2}\phi(0^-)$.

Note that we have

$$|\delta_\pm\rangle = |\delta_\star\rangle \pm \tfrac{1}{2}|j\rangle.$$

Lemma 2 *Introducing the form $\mathcal{J} \triangleq |\delta_+\rangle\langle\delta_+| - |\delta_-\rangle\langle\delta_-|$ on domain $AC(\mathbb{R}/\{0\})$, we have the following identity*

$$\mathcal{J} = |j\rangle\langle\delta_\star| + |\delta_\star\rangle\langle j|. \quad (1)$$

Proof. The right hand side is $\tfrac{1}{2}|\delta_+ - \delta_- \rangle\langle\delta_+ + \delta_-| + \tfrac{1}{2}|\delta_+ + \delta_- \rangle\langle\delta_+ - \delta_-|$ and expanding out gives the result. ■

2.2 Distributional First Order Derivatives

The differential operator $k_0 \equiv i\partial$ is symmetric on $W^{1,2}(\mathbb{R})$ and its closure is the generator of translations of wavefunctions along the real axis. This is not true when we try restrict to $W^{1,2}(\mathbb{R}/\{0\})$ due to the jump discontinuity at the origin. In fact, $i\partial$ is not now symmetric as we have the integration by parts formula

$$\langle\psi|\partial\phi\rangle + \langle\partial\psi|\phi\rangle = \psi^*\phi|_0^{0+} \equiv \langle\psi|\mathcal{J}\phi\rangle,$$

for $\phi, \psi \in W^{1,2}(\mathbb{R}/\{0\})$. Instead, let us introduce the distributional derivative operator

$$iD \triangleq i\partial + i|\delta_\star\rangle\langle j|. \quad (2)$$

A combination of the lemma and the integration-by-parts formula shows that iD is symmetric operator on $W^{1,2}(\mathbb{R}/\{0\})$. Note that $D\phi$ is then typically a functional on $W^{1,2}(\mathbb{R}/\{0\})$ and not a function.

2.3 Albeverio-Kurasov Construction

Albeverio and Kurasov study rank-one perturbations of self adjoint operators k_0 that are not semibounded, specifically they study formal of the form $k_0 + |\varphi\rangle\langle\varphi|$ where $\langle\varphi|$ is a bounded functional on $\text{dom}(k_0)$ though the perturbation need not be form bounded. They consider the restriction \tilde{k}_0 of k_0 to the dense domain $\text{dom}(\tilde{k}_0) = \{\psi \in \text{dom}(k_0) : \langle\varphi|\psi\rangle = 0\}$, and study the scales of Hilbert spaces associated with the adjoint \tilde{k}_0^\dagger .

In our case, $k_0 = i\partial$ with domain $W^{1,2}(\mathbb{R})$ and, since we consider δ -function perturbations, the restricted domain will be taken to be $V_0 = \{\psi \in W^{1,2}(\mathbb{R}) : \psi(0) = 0\}$. In this \tilde{k}_0^\dagger will be the operator $i\partial$ with domain $W^{1,2}(\mathbb{R}/\{0\})$.

2.4 Deficiency Subspaces

Let us introduce the following pair of vectors $\phi_\pm \in W^{1,2}(\mathbb{R}/\{0\})$:

$$\phi_\pm(t) = \mp i e^{\pm t} 1_{(0,\infty)}(\pm t). \quad (3)$$

An elementary calculation shows that $\partial|\phi_\pm\rangle = \mp|\phi_\pm\rangle$ and that $\langle\mathcal{J}|\phi_\pm\rangle = -i$, so that $iD|\phi_\pm\rangle = \mp i|\phi_\pm\rangle + |\delta_*\rangle$, or

$$|\phi_\pm\rangle = (iD \pm i)^{-1} |\delta_*\rangle.$$

More generally $iD_\sigma|\phi_\pm\rangle = \mp i|\phi_\pm\rangle + |\zeta\rangle$.

It is convenient to realize iD as being the adjoint \tilde{k}_0^\dagger to the operator \tilde{k}_0 as the restriction of $i\partial$ to the dense domain $V_0 = \{\psi \in W^{1,2}(\mathbb{R}) : \psi(0) = 0\}$. Note that V_0 is a Hilbert space with the Sobolev inner product. The deficiency subspaces $V_\pm = \ker(\tilde{k}_0^\dagger \pm i)$ are then both one-dimensional and spanned by the defect vectors ϕ_\pm respectively: $V_\pm = \{\mathbb{C}\phi_\pm\}$. The domain of the adjoint can then be written as

$$\text{dom}(\tilde{k}_0^\dagger) = V_0 \oplus_{1,2} V_+ \oplus_{1,2} V_- \quad (4)$$

where the three subspaces are mutually orthogonal with respect to the Sobolev space inner product. (See, for instance, theorem X.2 [16].)

The elements of $\text{dom}(\tilde{k}_0^\dagger)$ can then be represented as

$$|\psi\rangle = |\psi_0\rangle + c_+ |\phi_+\rangle + c_- |\phi_-\rangle$$

where $\psi_0 \in V_0$ and $c_\pm = \pm i\psi(0^\pm) \in \mathbb{C}$, the action of the adjoint is

$$\tilde{k}_0^\dagger |\psi\rangle = i\partial|\psi_0\rangle - ic_+ |\phi_+\rangle + ic_- |\phi_-\rangle.$$

We therefore represent iD by \tilde{k}_0^\dagger and to check consistency, we note that for $\psi \in \text{dom}(\tilde{k}_0^\dagger)$

$$\begin{aligned} iD|\psi\rangle &= i\partial|\psi_0\rangle + i\psi(0^+) (-i|\phi_+\rangle + |\delta_*\rangle) - i\psi(0^-) (i|\phi_-\rangle + |\delta_*\rangle) \\ &= i\partial|\psi\rangle + i\langle\mathcal{J}|\psi\rangle |\delta_*\rangle. \end{aligned}$$

For $\phi, \psi \in \text{dom}(\tilde{k}_0^\dagger)$, the associated boundary form $[\phi|\psi] := \langle \phi | \tilde{k}_0^\dagger \psi \rangle + \langle \tilde{k}_0^\dagger \phi | \psi \rangle$ will be $[\phi|\psi] = -i \langle \phi | \mathcal{I} \psi \rangle$. All self-adjoint extensions of k_0 can then be parameterized by a unimodular complex parameter s . We define $k_{0,s}$ as the restriction of \tilde{k}_0^\dagger to the domain

$$\text{dom}(\tilde{k}_{0,s}) := \left\{ \psi \in \text{dom}(\tilde{k}_0^\dagger) : \psi(0^+) = s\psi(0^-) \right\},$$

where the functions have an abrupt phase change s as we pass through the origin.

2.5 Singular Perturbations of $i\partial$

We return to the formal Hamiltonian $k = i\partial + E\delta$ which we now interpret as

$$k = iD + E|\delta_*\rangle\langle\delta_*|$$

The Hamiltonian can then be split up into continuous and singular components $k = k_{\text{ac}} + k_{\text{sing}}$ where $k_{\text{ac}} = i\partial$ and

$$k_{\text{sing}} = i|\delta_*\rangle\langle j| + E|\delta_*\rangle\langle\delta_*|.$$

Generally speaking, $k|\psi\rangle$ will be a functional for given $\psi \in \text{dom}(\tilde{k}_0^\dagger)$ except however when $k_{\text{sing}}|\psi\rangle = 0$ and the space of vector for which this vanishes defines the domain of k . This requires that $\{i\langle j| + E\langle\delta_*|\}\psi\rangle = 0$, or

$$i[\psi(0^+) - \psi(0^-)] + \frac{1}{2}E[\psi(0^+) + \psi(0^-)] = 0.$$

This can be written as the boundary condition

$$\psi(0^-) = \frac{1 - \frac{i}{2}E}{1 + \frac{i}{2}E}\psi(0^+). \quad (5)$$

The domain of k is therefore the set of $\psi \in W^{1,2}(\mathbb{R}/\{0\})$ satisfying (5). Note that the phase change $s = \frac{1 - \frac{i}{2}E}{1 + \frac{i}{2}E}$ is the Cayley transform of $\frac{1}{2}E$.

3 Second Quantization

We now consider the Hilbert space

$$\mathfrak{H} = \mathfrak{h} \otimes \Gamma(L_{\mathfrak{K}}^2(\mathbb{R}, dt))$$

where $\mathfrak{h}, \mathfrak{K}$ are fixed separable Hilbert spaces and $\Gamma(\cdot)$ is the bosonic Fock functor. A typical vector in \mathfrak{H} is $\Phi = (\Phi_n)$ where, for $n = 0, 1, 2, \dots$, we have that Φ_n is a $\mathfrak{h} \otimes \mathfrak{K}^{\otimes n}$ -valued function on \mathbb{R}^n symmetric in all its arguments and such that

$$\sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{R}^n} \|\Phi_n(t_1, \dots, t_n)\|^2 dt_1 \cdots dt_n < \infty.$$

We shall treat the case $\mathfrak{K} = \mathbb{C}$ initially for transparency.

3.1 Second quantization of iD

Fock space has the continuous tensor product decomposition $\Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \cong \Gamma(\mathfrak{h}_1) \oplus \Gamma(\mathfrak{h}_2)$ and we exploit the direct sum decomposition (4) to write

$$\Gamma\left(\text{dom}(k_0^\dagger)\right) \cong \Gamma_0 \otimes_{1,2} \Gamma_+ \otimes_{1,2} \Gamma_-, \quad (6)$$

where $\Gamma_\# = \Gamma(V_\#)$ for $\# = 0, +, -$, and the tensor product is with respect to the Sobolev inner product. Recall that every $\psi \in \text{dom}(\tilde{k}_0^\dagger)$ will have the decomposition $\psi = \psi_0 + i\psi(0^+)\phi_+ - i\psi(0^-)\phi_-$ and can define the usual exponential vectors $\varepsilon(\psi) = \sum_{n \geq 0} \frac{1}{n!} \otimes_{1,2}^n \psi$ and on this domain define the annihilation fields $A(\phi)$, with $\phi \in \text{dom}(\tilde{k}_0^\dagger)$, by

$$A(\phi)\varepsilon(\psi) = \langle \phi | \psi \rangle_{1,2} \varepsilon(\psi).$$

where $\langle \phi | \psi \rangle_{1,2} \equiv \langle \phi_0 | \psi_0 \rangle_{1,2} + \psi^*(0^+)\phi(0^+) + \psi^*(0^-)\phi(0^-)$.

Now the spaces V_\pm are both one-dimensional and so the Fock spaces $\Gamma(V_\pm)$ each correspond to the Hilbert space of an independent single mode harmonic oscillator and we take the respective annihilator operators to be

$$a_\pm \triangleq A(\pm i\phi_\pm). \quad (7)$$

With this convention, we have for $\phi \in \text{dom}(K_0^\dagger)$, $A(\phi) = A(\phi_0) + \phi(0^+)a_+ + \phi(0^-)a_-$. In particular, since $\langle \pm i\phi_\pm | \psi \rangle_{1,2} = \phi(0^\pm)$, we have

$$a_\pm \varepsilon(\psi) = \psi(0^\pm) \varepsilon(\psi).$$

It is convenient to introduce

$$a_\star \triangleq \frac{1}{2}(a_+ + a_-)$$

The second quantization of $\tilde{k}_0^\dagger = iD$ is then given by $K_0 = K_{0,\text{ac}} + K_{0,\text{sing}}$ where

$$K_{0,\text{ac}} = d\Gamma(i\partial), \quad K_{0,\text{sing}} = id\Gamma(|\delta_\star\rangle\langle j|) = ia_\star^\dagger(a_+ - a_-). \quad (8)$$

3.2 Singular Perturbations of $K_{\text{ac}} = d\Gamma(i\partial)$

We now consider the perturbed Hamiltonian $K = K_0 + \Upsilon$ where

$$\Upsilon = E_{11}a_\star^\dagger a_\star + E_{10}a_\star^\dagger + E_{01}a_\star + E_{00} \quad (9)$$

where the $E_{\alpha\beta}$ are operators on \mathfrak{h} with $(E_{\alpha\beta})^\dagger = E_{\beta\alpha}$. The Hamiltonian may then be decomposed into continuous and singular parts as

$$K_{\text{ac}} + E_{00} + E_{01}a_\star + a_\star^\dagger(ia_+ - ia_- + E_{11}a_\star + E_{10}).$$

Let us introduce the following operators on the system space

$$S = \frac{1 - \frac{i}{2}E_{11}}{1 + \frac{i}{2}E_{11}}, \quad L = -\frac{i}{1 + \frac{i}{2}E_{11}}E_{10}, \quad H = E_{00} + \frac{1}{2}E_{01} \text{Im} \left\{ \frac{1}{1 + \frac{i}{2}E_{11}} \right\} E_{10}. \quad (10)$$

Theorem 3 *The domain of the self-adjoint extension corresponding to the operator $K = K_0 + \Upsilon$ is the set of vectors $\Phi \in \mathfrak{h} \otimes \text{dom}(K_0)$ satisfying the boundary conditions*

$$a_- \Phi = S a_+ \Phi + L, \quad (11)$$

and on this domain we have

$$-iH_{\text{total}}\Phi = \left(-iK_{ac} - \frac{1}{2}L^\dagger L - iH\right)\Phi - L^\dagger S a_+ \Phi. \quad (12)$$

Proof. Our strategy is basically the same as in the one-particle case - we choose the domain of the operator to consist of those vectors Φ such that $(ia_+ - ia_- + E_{11}a_\star + E_{10})\Phi = 0$. This is equivalent to the boundary condition (11). For vectors on this domain, we may substitute $a_- \Phi$ for $a_+ \Phi$ using the boundary condition and this gives the desired result. ■

3.3 Multiple Field Modes

We now consider the more general case where $\mathfrak{K} = \mathbb{C}^n$. Let $\{e_j : j = 1, \dots, n\}$ be an orthonormal basis for \mathfrak{K} . The one-particle space is now $L^2_{\mathbb{C}^n}(\mathbb{R}, dt) = \mathbb{C}^n \otimes L^2(\mathbb{R}, dt)$ and we replace the decomposition (6) by $\Gamma(\mathbb{C}^n \otimes \text{dom}(\tilde{k}_0^\dagger)) = \Gamma_0^{(n)} \otimes_{1,2} \Gamma_+^{(n)} \otimes_{1,2} \Gamma_-^{(n)}$ where $\Gamma_\#^{(n)} = \Gamma(\mathbb{C}^n \otimes V_\#)$. The defect vectors for the ampliation of \tilde{k}_0 to $\mathbb{C}^n \otimes L^2(\mathbb{R}, dt)$ may now be fixed as $e_j \otimes \phi_\pm$ and so the deficiency indices are now (n, n) .

Proceeding as before, we introduce the independent annihilators

$$a_{j,\pm} \triangleq A(\pm i e_j \otimes \phi_\pm)$$

and the second quantization of the ampliation of \tilde{k}_0^\dagger is $K_0 = K_{0,\text{ac}} + K_{0,\text{sing}}$ with

$$K_{0,\text{ac}} = d\Gamma(i\partial), \quad K_{0,\text{sing}} = i a_{j,\star}^\dagger (a_{j,+} - a_{j,-}). \quad (13)$$

(A summation over the range $1, \dots, n$ is implied for repeated Latin indices!)

Let us also set $a_{0,\pm} \equiv 1$. As perturbation we consider the singular term

$$\Upsilon = E_{\alpha\beta} a_\alpha^\dagger a_\beta. \quad (14)$$

(Here we understand repeated Greek indices as implying a sum over $0, 1, \dots, n$.) Again the $(n+1)^2$ operators $E_{\alpha\beta}$ are operators on $\mathfrak{B}(\mathfrak{h})$ and we require that $(E_{\alpha\beta})^\dagger = E_{\beta\alpha}$. It is convenient to assemble them into a matrix \mathbf{E} . More generally we consider the class of matrices

$$\mathbf{X} = \begin{pmatrix} X_{00} & X_{0\ell} \\ X_{\ell 0} & X_{\ell\ell} \end{pmatrix} \in \mathfrak{B}((\mathbb{C} \oplus \mathfrak{K}) \otimes \mathfrak{h})$$

where $X_{00} \in \mathfrak{B}(\mathfrak{h})$, $X_{0\ell} \in \mathfrak{B}(\mathfrak{h}, \mathfrak{K} \otimes \mathfrak{h})$, $X_{\ell 0} \in \mathfrak{B}(\mathfrak{K} \otimes \mathfrak{h}, \mathfrak{h})$ and $X_{\ell\ell} \in \mathfrak{B}(\mathfrak{K} \otimes \mathfrak{h})$. That is $X_{0\ell}$ is the row vector (X_{01}, \dots, X_{0n}) and $X_{\ell 0}$ is the column vector $(X_{01}, \dots, X_{0n})^T$ while $X_{\ell\ell} = (X_{ij})$.

Let us also introduce the special matrix $\mathbf{\Pi}$ projecting onto the subspace $\mathfrak{K} \otimes \mathfrak{h}$ of $(\mathbb{C} \oplus \mathfrak{K}) \otimes \mathfrak{h} \equiv \mathfrak{h} \oplus (\mathfrak{K} \otimes \mathfrak{h})$, that is,

$$\mathbf{\Pi} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Its coefficients are the Evans-Hudson delta

$$\hat{\delta}_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta \in \{1, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Given a self-adjoint $\mathfrak{B}(\mathfrak{h})$ -valued matrix \mathbf{E} as above, we recall some related matrices, originally introduced in [13], starting with the matrix \mathbf{G} defined by the identity $\mathbf{G} = -i\mathbf{E} - \frac{i}{2}\mathbf{E}\mathbf{\Pi}\mathbf{G}$,

$$\begin{aligned} (\text{It}\bar{o}) \ \mathbf{G} &\triangleq -i \left(\mathbf{1} + \frac{i}{2}\mathbf{E}\mathbf{\Pi} \right)^{-1} \mathbf{E} = \begin{pmatrix} -\frac{1}{2}L^\dagger L - iH & -L^\dagger S \\ L & S - 1 \end{pmatrix}, \\ (\text{Model}) \ \mathbf{V} &\triangleq \mathbf{G} + \mathbf{\Pi} = \begin{pmatrix} -\frac{1}{2}L^\dagger L - iH & -L^\dagger S \\ L & S \end{pmatrix}, \\ (\text{Galilean}) \ \mathbf{M} &\triangleq (\mathbf{1} + \mathbf{\Pi}\mathbf{G}) = \begin{pmatrix} 1 & 0 \\ L & S \end{pmatrix}, \\ (\text{Dressing}) \ \mathbf{F} &\triangleq \left(\mathbf{1} + \frac{1}{2}\mathbf{\Pi}\mathbf{G} \right) = \left(\mathbf{1} + \frac{i}{2}\mathbf{\Pi}\mathbf{E} \right)^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2}L & \frac{1}{2}(S+1) \end{pmatrix}. \end{aligned}$$

Note the identity

$$\mathbf{G} = -i\mathbf{E}\mathbf{F} \tag{15}$$

and that we encounter the operators

$$\begin{aligned} S &= \left(1 - \frac{i}{2}E_{\ell\ell} \right) \left(1 + \frac{i}{2}E_{\ell\ell} \right)^{-1} \in \mathfrak{B}(\mathfrak{K} \otimes \mathfrak{h}), \\ L &= -i \left(1 + \frac{i}{2}E_{\ell\ell} \right)^{-1} E_{\ell 0} \in \mathfrak{B}(\mathfrak{K} \otimes \mathfrak{h}, \mathfrak{h}), \\ H &= E_{00} + \frac{1}{2} \text{Im } E_{0\ell} \left(1 + \frac{i}{2}E_{\ell\ell} \right)^{-1} E_{\ell 0}. \end{aligned} \tag{16}$$

With these conventions we may write

$$K_{\text{sing}} = i\mathbf{a}_\star^\dagger \mathbf{\Pi} (\mathbf{a}_+ - \mathbf{a}_-), \quad \Upsilon = \mathbf{a}_\star^\dagger \mathbf{E} \mathbf{a}_\star,$$

where

$$\mathbf{a}_\pm \triangleq \begin{pmatrix} 1 \\ a_{1,\pm} \\ \vdots \\ a_{n,\pm} \end{pmatrix}, \quad \mathbf{a}_\star = \frac{1}{2}(\mathbf{a}_+ + \mathbf{a}_-).$$

The Hamiltonian is then

$$K = d\Gamma(i\partial) + \mathbf{a}_\star^\dagger \{i\Pi(\mathbf{a}_+ - \mathbf{a}_-) + \mathbf{E}\mathbf{a}_\star\}$$

and the domain is the set of vectors Φ such that $\Pi\{i(\mathbf{a}_+ - \mathbf{a}_-) + \mathbf{E}\mathbf{a}_\star\}\Phi = 0$. This boundary condition ensures that all the $a_{j,\star}^\dagger$ terms vanish ($j = 1, \dots, n$) and may be reformulated as

$$\Pi(\mathbf{E} - 2i\Pi)\mathbf{a}_+\Phi + \Pi(\mathbf{E} + 2i\Pi)\mathbf{a}_-\Phi = 0$$

or

$$a_{j,-}\Phi = S_{jk}a_{k,+}\Phi + L_j\Phi$$

with $S \equiv S_{jk} \otimes |e_j\rangle\langle e_k|$ and $L \equiv L_j \otimes |e_j\rangle$. We have trivially that $a_{0,-} = a_{0,+}$ and we may include this in the boundary condition to write

$$\mathbf{a}_+\Phi = \mathbf{M}\mathbf{a}_-\Phi. \quad (17)$$

Using the boundary condition, we may write

$$\begin{aligned} K\Phi &= K_{\text{ac}}\Phi + \frac{1}{2}E_{0\alpha}\{a_{\alpha,+} + a_{\alpha,-}\}\Phi \\ &= K_{\text{ac}}\Phi + \frac{1}{2}E_{0\alpha}\{\delta_{\alpha\beta} + M_{\alpha\beta}\}a_{\beta,+}\Phi \end{aligned}$$

however $\frac{1}{2}\mathbf{E}(\mathbf{1} + \mathbf{M}) = \mathbf{E}(\mathbf{1} + \frac{1}{2}\Pi\mathbf{G}) = \mathbf{E}\mathbf{F} = i\mathbf{G}$. With this we may write the action of the Hamiltonian on vectors satisfying the boundary conditions as

$$K\Phi = K_{\text{ac}}\Phi + iG_{0\alpha}a_{\beta,+}\Phi.$$

We may summarize our findings using the model matrix \mathbf{V} in the following theorem.

Theorem 4 *The Hamiltonian associated to $K = K_0 + \Upsilon$ with perturbation (14) is defined on the domain $\Gamma\left(\text{dom}(\tilde{k}_0^\dagger)\right)$ satisfying the boundary condition $\mathbf{a}_+\Phi = \mathbf{V}\mathbf{a}_-\Phi$. Its action on this domain, and the boundary condition, are given by*

$$\begin{aligned} -iK\Phi &= \left(V_{00} + V_{0k}a_{k,+} - i\tilde{K}_0\right)\Phi = \left(V_{0\beta}a_{\beta}(0^+) - i\tilde{K}_0\right)\Phi, \\ a_{j,-}\Phi &= V_{j0}\Phi + V_{jk}a_{k,+}\Phi = V_{j\beta}a_{\beta,+}\Phi. \end{aligned}$$

4 Gauge freedom

A complex number κ_+ with strictly positive real part will be referred to as a *complex damping constant*. For convenience we shall normalize complex damping constants as

$$\kappa_\pm = \frac{1}{2} \pm i\sigma,$$

where σ is real. For a fixed complex damping κ we then define a functional $\zeta = \zeta_\kappa$ by

$$|\zeta\rangle = \kappa_+ |\delta_+\rangle + \kappa_- |\delta_-\rangle \equiv \gamma |\delta_\star\rangle + i\sigma |j\rangle. \quad (18)$$

We then we have the following local identity generalizing (1)

$$\mathcal{J} = |j\rangle \langle \zeta| + |\zeta\rangle \langle j|, \quad (19)$$

as the $|j\rangle \langle j|$ terms cancel. This allows us to construct a more general class of self-adjoint extensions of the restriction of $i\partial$. Let σ be the imaginary part of the complex damping κ_+ and define

$$iD_\sigma \triangleq i\partial + i|\delta_\star\rangle \langle j| - \sigma |j\rangle \langle j| \equiv i\partial + i|\zeta\rangle \langle j|$$

with ζ as above. It follows that iD_σ is likewise a symmetric operator on $W^{1,2}(\mathbb{R}/\{0\})$.

The formal Hamiltonian $k = i\partial + E\delta$ may be alternatively interpreted as

$$k_\sigma = iD_\sigma + E|\zeta\rangle \langle \zeta|,$$

or $k_\sigma = i\partial + i|\zeta\rangle \langle j| + E|\zeta\rangle \langle \zeta|$. We may follow the same argument as before and arrange for the singular component to vanish by imposing the boundary conditions: this time the condition (5) is modified to $\psi(0^-) = s_\sigma \psi(0^+)$ where we now have the phase $s_\sigma = \frac{1 - i\kappa_- E}{1 + i\kappa_+ E}$.

This can be lifted immediately to the second quantization. For $\mathfrak{K} = \mathbb{C}$, we set $\mathfrak{a} = \kappa_- a_+ + \kappa_+ a_-$. We make the corresponding replacements: $K_{0,\text{sing}}(\sigma) = i\mathfrak{a}^\dagger (a_+ - a_-)$ and $\Upsilon(\sigma) = E_{11}\mathfrak{a}^\dagger \mathfrak{a} + E_{10}\mathfrak{a}^\dagger + E_{01}\mathfrak{a} + E_{00}$. The boundary conditions arise from requiring the \mathfrak{a}^\dagger terms to vanish and after similar algebra to before we arrive at the restatement of the first theorem with the modified operators

$$\begin{aligned} S(\sigma) &= \frac{1 - i\kappa_- E_{11}}{1 + i\kappa_+ E_{11}}, \quad L(\sigma) = -\frac{i}{1 + i\kappa_+ E_{11}} E_{10}, \\ H(\sigma) &= E_{00} + \text{Im} \left\{ E_{01} \frac{\kappa_+}{1 + i\kappa_+ E_{11}} E_{10} \right\}. \end{aligned} \quad (20)$$

Identical expression where obtained for the singular limit of finite time correlated Bose field in [12], theorem 8.1 equation (8.11).

The multiple field mode version of this is to introduce in place of $\mathbf{a}_\star = \frac{1}{2}(\mathbf{a}_+ + \mathbf{a}_-)$ the vector

$$\mathfrak{a} = \left(\frac{1}{2}\mathbf{1} + i\mathbf{Z}\right)\mathbf{a}_+ + \left(\frac{1}{2}\mathbf{1} - i\mathbf{Z}\right)\mathbf{a}_-$$

where $\mathbf{Z} = \begin{pmatrix} 0 & 0 \\ 0 & Z_{\ell\ell} \end{pmatrix}$ is a hermitean matrix with complex entries. We then obtain a straightforward modification of the second theorem with new coefficients

$\mathbf{G}(\mathbf{E}, \mathbf{Z}) = -i \left(\mathbf{1} + i\mathbf{E} \left(\frac{1}{2}\mathbf{\Pi} + i\mathbf{Z} \right) \right)^{-1} \mathbf{E}$. This general type was first introduced in [13].

In general, the parameters (Z_{jk}) may be termed “gauge parameters” and by choosing a different set of parameters we obtain a different self-adjoint extension. This type freedom/ambiguity is well-known and is ultimately a question of fixing the desired physical model and cannot arise from purely mathematical arguments alone [16].

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